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PLASTIC RUPTURE LINES AT THE TIP OF A CRACK

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An exact solution of the problem of the initial development of plastic strains near the tips of cleavage (normal) cracks is given. Under plane strain conditions the plastic strains are assumed to be concentrated along two narrow slip bands issuing symmetrically from the tip of the crack at some angle α to its continuation. Such a shape of the plastic domain can be considered as a certain approximation to the "spread" plastic zone; for some materials it possibly corresponds more closely to the essence of the phenomenon. The length of the slip line and the angle α are determined (for instance, the angle α turns out to be equal 72°). The influence of a side load on the plastic domain can be also analyzed successfully in this model. It is shown, in particular, that the opening at the tip of the crack depends essentially on the side load; therefore, the extensively used criterion C. O. D. (the crack opening displacement) is not a local criterion even in the case of an arbitrarily small plastic zone. The technique of integral transforms and the Wiener-Hopf method are used to solve the problem.

1. Introduction. Let us consider a homogeneous isotropic body with arbitrary cleavage cracks. We shall assume the body material to be ideally elastic-plastic and to satisfy the Tresca-Saint Venant plasticity condition, and the strains to be small. The cracks will be represented as zero-thickness mathematical slits. Hence, for arbitrarily small external loads near the crack contour, a plastic domain will originate.

For sufficiently small external loads, the characteristic linear dimension of the plastic zone will be small compared to the characteristic linear dimension of the body and the cracks. In this case, the formulation of the asymptotic problem on the fine structure of the tip of a crack [1] is valid; the crack can be considered as a semi-infinite slit along the negative x semiaxis in the xy-plane, free of external loads, with an additional loading condition determined from the solution of the purely elastic problem imposed at infinity. The solution of this plane problem describes the field of stresses and strains in the neighborhood of any point of a smooth crack contour to within the accuracy of a factor (the stress intensity factor K_I). A number of problems were, for example, examined in such a formulation in [1 — 5].

It is assumed below that the plastic strains are concentrated along narrow rectilinear slip bands issuing from the tip of the crack. The length of the plastic bands is evidently enlarged with the increase of the external load and should be determined during solution of the problem. We consider the number of slip bands to equal two (Fig. 1); they are symmetric relative to the plane of the crack and are at angle α to its continuation (the problem is considered locally symmetric). The length of the slip band in the xy-plane can be taken as the unit of the linear scale, and thereby set equal to unity without loss of generality.

Such a formulation of the problem can be considered, on the one hand, as approximate and descriptive of the spread plastic zone near the tip of the crack. On the other hand, some results indicate that plastic strains in certain materials have the tendency to concentrate in narrow slip bands (which are lines of discontinuity of the displacement). In this realm the tests on thin low-carbon steel sheets carried out by Dugdale, who discovered and computed such slip bands in the continuation of a crack [6], are classical. The paper [7], in which two more side plasticity bands in addition to the Dugdale band are detected for a further increase in the load, is a further essential development of these views. Let us note that if the Dugdale band corresponds to the plastic flow typical for the plane state of stress (i.e. for ideally thin plates), then the two Leonov-Vitvitskii-Iarema side bands [7] correspond to the plastic flow characteristic for plane strain (i.e. for infinitely thick plates).

Formulation of the problem with side plasticity bands moreover permits explanation and computation of the effect of biaxiality of the state of stress which it is difficult to conceive in other approaches (perhaps even more exact ones).

2. Derivation of the Wiener-Hopf equation. Let us write the boundary conditions for the considered problem in the case of two plasticity bands (see Fig. 1) by assuming that the stresses vanish at infinity in a given manner:

$$\theta = \pm\pi, \quad \sigma_\theta = \tau_{r\theta} = 0 \tag{2.1}$$

$$\theta = 0, \ u_\theta = 0, \ \tau_{r\theta} = 0 \tag{2.2}$$

$$\theta = \pm\alpha, \ [\sigma_\theta] = [\tau_{r\theta}] = 0, \ [u_\theta] = 0 \tag{2.3}$$

$$\theta = \pm\alpha, \begin{cases} 0 < r < 1, & \tau_{r\theta} = \tau_s \\ \infty > r > 1, & [\sigma_r] = 0 \end{cases} \tag{2.4}$$

$$\theta = 0, \ r \to \infty, \ \sigma_r = \sigma_\theta = \frac{K_{\rm I}}{\sqrt{2\pi r}} \tag{2.5}$$

The square brackets denote here a jump in the quantity enclosed in the brackets during passage through the line of discontinuity, τ_s is the shear yield point, and $K_{\rm I}$ is the stress intensity factor. Conditions (2.3) and (2.4) mean that in the side plasticity bands a discontinuity of only the tangential component of the displacement is admitted, while the shear stress equals the yield point. The symmetry condition (2.2) permits us to restrict the examination to the upper half-plane $0 < \theta < \pi$ for solving the problem.

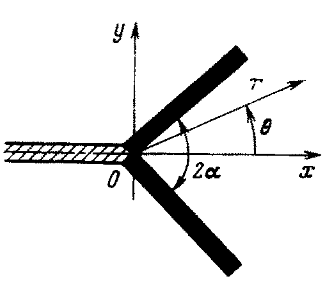

Fig. 1

We apply the Mellin integral transform (p is a complex parameter)

$$f(p) = \int_0^\infty f(r)\, r^p\, dr$$

to the statics equations and the compatibility condition. Consequently, we obtain the following fourth order differential equation for the function $\bar{\sigma}_\theta(p, \theta)$:

$$\frac{d^4\bar{\sigma}_\theta}{d\theta^4}+[(p+1)^2+(p-1)^2]\frac{d^2\bar{\sigma}_\theta}{d\theta^2}+(p+1)^2(p-1)^2\bar{\sigma}_\theta=0 \qquad (2.6)$$

The functions $\bar{\sigma}_r$, $\bar{\tau}_{r\theta}$ are expressed in terms of $\bar{\sigma}_\theta$ thus:

$$\bar{\tau}_{r\theta}=\frac{1}{p-1}\frac{d\bar{\sigma}_\theta}{d\theta}, \quad p\bar{\sigma}_r=\frac{1}{p-1}\frac{d^2\bar{\sigma}_\theta}{d\theta^2}-\bar{\sigma}_\theta \qquad (2.7)$$

The solution of (2.6) is

$$\bar{\sigma}_\theta(p,\theta)=A_1(p)\cos(p+1)\theta+A_2(p)\cos(p-1)\theta+ A_3(p)\sin(p+1)\theta+A_4(p)\sin(p-1)\theta \quad \text{for } 0\leqslant\theta\leqslant\alpha \qquad (2.8)$$

$$\bar{\sigma}_\theta(p,\theta)=B_1(p)\cos(p+1)(\theta-\pi)+B_2(p)\cos(p-1)(\theta-\pi)+ B_3(p)\sin(p+1)(\theta-\pi)+B_4(p)\sin(p-1)(\theta-\pi) \quad \text{for } \alpha\leqslant\theta\leqslant\pi \qquad (2.9)$$

Here $A_1, \ldots, A_4, B_1, \ldots, B_4$ are unknown functions of the parameter p which are determined from the boundary conditions. Any seven of these are expressed in terms of one unknown function by using seven "through" boundary conditions (2.1) — (2.3) transformed with respect to r. Noting that according to Hooke's law

$$\frac{\partial\bar{u}_r}{\partial r}=\frac{1+\nu}{E}[(1-\nu)\bar{\sigma}_r-\nu\bar{\sigma}_\theta] \qquad (2.10)$$

$$\frac{\partial\bar{u}_\theta}{\partial r}=\frac{1+\nu}{E(p+1)}\left[2p\bar{\tau}_{\theta r}+(1-\nu)\frac{d\bar{\sigma}_r}{d\theta}-\nu\frac{d\bar{\sigma}_\theta}{d\theta}\right]$$

(E is Young's modulus and ν is the Poisson's ratio, we find the following equations from the transformed boundary conditions (2.1) — (2.3) by using (2.8) — (2.10):

$$A_3=A_4=0,\ B_1+B_2=0,\ B_3(p+1)+B_4(p-1)=0$$

$$A_1\cos(p+1)\alpha+A_2\cos(p-1)\alpha=B_1\cos(p+1)(\alpha-\pi)+B_2\cos(p-1)(\alpha-\pi)+B_3\sin(p+1)(\alpha-\pi)+B_4\sin(p-1)(\alpha-\pi)$$

$$A_1(p+1)\sin(p+1)\alpha+A_2(p-1)\sin(p-1)\alpha= B_1(p+1)\sin(p+1)(\alpha-\pi)+B_2(p-1)\sin(p-1)(\alpha-\pi)-B_3(p+1)\cos(p+1)(\alpha-\pi)-B_4(p-1)\cos(p-1)(\alpha-\pi)$$

$$A_1(p+1)^3\sin(p+1)\alpha+A_2(p-1)^3\sin(p-1)\alpha=B_1(p+1)^3\sin(p+1)(\alpha-\pi)+B_2(p-1)^3\sin(p-1)(\alpha-\pi)-B_3(p+1)^3\cos(p+1)(\alpha-\pi)-B_4(p-1)^3\cos(p-1)(\alpha-\pi)$$

We write the solution of this system of equations as

$$A_1(p)=D(p)[p\cos p\alpha\sin\alpha-\sin p(\pi-\alpha)\cos(p\pi-\alpha)] \qquad (2.11)$$

$$A_2(p)=-D(p)[p\cos p\alpha\sin\alpha-\sin p(\pi-\alpha)\cos(p\pi+\alpha)]$$

$$B_1(p)=-D(p)\cos p\pi[p\cos p\alpha\sin\alpha+\sin p\alpha\cos\alpha]$$

$$B_4(p)=-\frac{p+1}{p-1}D(p)(p-1)\sin\alpha\sin p\pi\cos p\alpha$$

($D(p)$ is an unknown function of p). We introduce the following functions:

$$\Phi^-(p) = \int_0^1 [\sigma_r(r, \alpha)] r^p \, dr, \quad \Phi^+(p) = \int_1^\infty \tau_{r\theta}(r, \alpha) r^p \, dr \tag{2.12}$$

Here the function $\Phi^-(p)$ is evidently analytic in the half-plane $\mathrm{Re}\, p > -1$, and the function $\Phi^+(p)$ is analytic in the half-plane $\mathrm{Re}\, p < -{}^1/_2$ on the basis of condition (2.5). By using the functions introduced, the boundary conditions (2.4) are written as

$$\theta = \alpha, \quad \overline{[\sigma_r]} = \Phi^-(p), \quad \overline{\tau}_{r\theta} = \Phi^+(p) + \frac{\tau_s}{p+1} \tag{2.13}$$

The conditions (2.13) can be written by using (2.8), (2.9) and (2.11) as

$$\Phi^-(p) = -\frac{2\sin 2p\pi}{p-1} D(p), \quad \Phi^+(p) + \frac{\tau_s}{p+1} = \frac{\sin^2 p\pi}{p-1} G(p) D(p) \tag{2.14}$$

$$G(p) = \frac{1}{2\sin^2 p\pi} \{(p \sin 2\alpha + \sin 2p\alpha)[\sin 2p(\pi - \alpha) - p\sin 2\alpha] +$$
$$2(\cos 2p\alpha - \cos 2\alpha)[\sin^2 p(\pi - \alpha) - p^2 \sin^2 \alpha]\}$$

Eliminating the function $D(p)$ from the two relationships in (2.14), we obtain the following Wiener-Hopf functional equation:

$$\Phi^+(p) + \frac{\tau_s}{p+1} = -\frac{1}{4} \operatorname{tg} \pi p G(p) \Phi^-(p) \tag{2.15}$$

3. Solution of the boundary value problem. The functional equation (2.15) holds in the strip $-1 < \mathrm{Re}\, p < -{}^1/_2$, $-\infty < \mathrm{Im}\, p < +\infty$. The function $G(p)$ in this equation possesses the following properties:

a) the function $G(p)$ is meromorphic, its second order poles are the points $p = \pm 1, \pm 2, \ldots$ with the exception, perhaps, of roots of the equation $p \operatorname{tg} \alpha = -\operatorname{tg} p\alpha$;

b) the function has neither poles nor zeros anywhere on the line $p = -{}^1/_2 + it$ $(-\infty < t < +\infty)$;

c) $p \to \infty$ along the line $p = -{}^1/_2 + it$, the function $G(p)$ tends to unity.

Let us examine the contour L consisting of the line $p = -{}^1/_2 + it$ and a left semicircle of small radius with center at the point $p = -{}^1/_2$ in the p plane (Fig. 2). The direction of traversing the contour L agrees with the direction of the imaginary axis. The domains to the left and right of the contour L will be denoted by D_+ and D_-, respectively. The function $G(p)$ can be represented on the contour L as

$$G(p) = \frac{G^+(p)}{G^-(p)} \quad (p \in L) \tag{3.1}$$

$$\exp \frac{1}{2\pi i} \int_L \frac{\ln G(t)}{t-p} dt = \begin{cases} G^+(p) & (p \in D_+) \\ G^-(p) & (p \in D_-) \end{cases} \quad (G^\pm(\pm\infty) = 1) \tag{3.2}$$

Here $G^+(p)$ and $G^-(p)$ are entire functions, analytic and without zeros in the domains D_+ and D_-, respectively. As $|t| \to \infty$, the function $\ln G(t)$ decreases exponentially on L, hence the integral (3.2) converges rapidly.

We will use the following known representation (see [8]):

$$p \operatorname{ctg} \pi p = -K^+(p) K^-(p) \tag{3.3}$$

$$K^+(p) = \frac{\Gamma(1-p)}{\Gamma(-{}^1/_2 - p)}, \quad K^-(p) = \frac{\Gamma(1+p)}{\Gamma({}^3/_2 + p)} \tag{3.4}$$

The function $K^+(p)$ is analytic and has zeros for $\operatorname{Re} p < -{}^1/_2$, while the function $K^-(p)$ is analytic and has no zeros for $\operatorname{Re} p > -1$. Moreover, we have according to the Stirling formula

$$K^+(p) = (-p)^{3/2} + O(1), \quad K^-(p) = (+p)^{-1/2} + O(1), \qquad p \to \infty \tag{3.5}$$

By using the factorizations (3.1) and (3.3) we can write the functional equation (2.15) as:

$$\frac{\Phi^+(p)\,K^+(p)}{pG^+(p)} + \frac{\tau_s K^+(p)}{p\,(p+1)\,G^+(p)} = \frac{\Phi^-(p)}{4G^-(p)\,K^-(p)} \quad (p \in L) \tag{3.6}$$

We now use the following representation :

$$\frac{K^+(p)}{p\,(p+1)\,G^+(p)} = \frac{1}{p+1}\left[\frac{K^+(p)}{pG^+(p)} + \frac{K^+(-1)}{G^+(-1)}\right] - \frac{K^+(-1)}{(p+1)\,G^+(-1)} \tag{3.7}$$

Substituting (3.7) into (3.6), we obtain

$$\frac{\Phi^+(p)\,K^+(p)}{pG^+(p)} + \frac{\tau_s}{p+1}\left[\frac{K^+(p)}{pG^+(p)} + \frac{K^+(-1)}{G^+(-1)}\right] = \frac{\tau_s K^+(-1)}{(p+1)\,G^+(-1)} + \frac{\Phi^-(p)}{4G^-(p)\,K^-(p)} \quad (p \in L) \tag{3.8}$$

The left side of this equality is analytic in D_+ and the right side in D_-. On the basis of the principle of analytic continuation, they equal the same function which is analytic in the whole plane. In order to find this single analytic function it is necessary to study the behavior of the sought functions $\Phi^-(p)$ and $\Phi^+(p)$ as $p \to \infty$. To do this, we consider a stress concentration (*) characterized by elastic asymptotics for cracks with a plastic filler [1] to exist in the general case at the head of the slip line. This elastic asymptotics (local or hyperfine structure) is defined completely by the single stress intensity factor k_{II}. The critical value of this factor k_{IIc} (slip ductility) determines the resistance of the material to development of slip surfaces therein. We consider the quantity k_{IIc} as a specified constant of the material. When such a resistance is negligible, it can be considered that $k_{IIc} = 0$ and the stresses will be bounded at the end of the slip band in this particular case. According to [9], the following asymptotics holds:

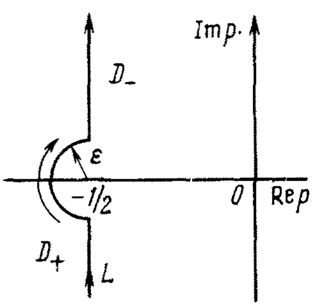

Fig. 2

$$\Phi^-(p) = \frac{2\sqrt{2}\,k_{II}}{\sqrt{p}}, \quad \Phi^+(p) = -\frac{k_{II}}{\sqrt{-2p}}, \quad p \to \infty \tag{3.9}$$

On the basis of (3.9), (3.2) and (3.5), the single analytic function in (3.8) tends to the constant $k_{II}/\sqrt{2}$ as $p \to \infty$, and therefore, it equals this constant identically in the whole p plane according to the Liouville theorem.

Therefore, the solution of the boundary value problem can be written as

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*) This concentration may be due to the accumulation of dislocations in the form of a Cottrell cloud (it can be essential for materials of the low-carbon steel type, for example).

$$\Phi^+(p) = -\frac{\tau_s pG^+(p)}{(p+1)K^+(p)}\left[\frac{K^+(p)}{pG^+(p)} + \frac{K^+(-1)}{G^+(-1)}\right] - \frac{k_{II}pG^+(p)}{\sqrt{2}K^+(p)} \qquad (3.10)$$

$$\Phi^-(p) = -\frac{4\tau_s G^-(p)K^-(p)K^+(-1)}{(p+1)G^+(-1)} - 2\sqrt{2}k_{II}G^-(p)K^-(p)$$

Hence, determining the function $D(p)$ by using (2.14), we find the Mellin transform of the stresses and the stresses themselves after inverting the transform.

4. Analysis of the solution. The unknown parameter k_{II}, which should be determined from the condition (2.5) at infinity, enters into the solution (3.10). It is still more convenient to use the second formula in (2.12) and the known asymptotics for cleavage cracks

$$\theta = \alpha, \quad r \to \infty, \quad \tau_{r\theta} = \frac{K_I}{2\sqrt{2\pi r}}\sin\alpha\cos\frac{\alpha}{2} \qquad (4.1)$$

By using a theorem of Abel-type it is hence easy to find

$$\Phi^+(p) = -\frac{K_I \sin\alpha\cos\alpha/2}{2\sqrt{2\pi}\,(p+{}^1/_2)}, \quad p \to -\frac{1}{2} \qquad (4.2)$$

According to (3.4), we have

$$K^+(p) = -\frac{\Gamma({}^3/_2)(p+{}^1/_2)}{\Gamma({}^1/_2)}\sqrt{\pi}, \quad p \to -\frac{1}{2} \qquad (4.3)$$

The first formula of (3.10) yields

$$\Phi^+(p) = \frac{-k_{II}G^+(-{}^1/_2)}{2\sqrt{2}\,\Gamma({}^3/_2)(p+{}^1/_2)} - \frac{\tau_s G^+(-{}^1/_2)}{\Gamma({}^3/_2)(p+{}^1/_2)} \times \qquad (4.4)$$
$$\left[\frac{K^+(-1)}{G^+(-1)} - \frac{2K^+(-{}^1/_2)}{G^+(-{}^1/_2)}\right], \quad p \to -\frac{1}{2}$$

Equating (4.2) and (4.4), we find

$$-k_{II} = -K_I\frac{\Gamma({}^3/_2)}{G^+(-{}^1/_2)\sqrt{\pi}}\sin\alpha\cos\frac{\alpha}{2} + \qquad (4.5)$$
$$2\sqrt{2}\tau_s\left[\frac{K^+(-1)}{G^+(-1)} - \frac{2K^+(-{}^1/_2)}{G^+(-{}^1/_2)}\right]$$

Passing to the dimensional variables and equating the local intensity factor to the constant k_{IIc} of the material, we obtain an equation for determining the length d of the slip band

$$d^{1/2} = \frac{-k_{IIc} + K_I \sin\alpha\cos\alpha/2\,[2G^+(-{}^1/_2)]^{-1}}{2\sqrt{2}\tau_s K^+(-1)\,[G^+(-1)]^{-1}} \qquad (4.6)$$

In particular, if the stress concentration at the head of the slip band is absent, i.e. $k_{IIc} = 0$, the formula is simplified

$$\frac{\tau_s d^{1/2}}{K_I} = \frac{\sqrt{\pi}}{4\sqrt{2}}\,\frac{\sin\alpha\cos\alpha/2\,G^+(-1)}{G^+(-{}^1/_2)} \qquad (4.7)$$

Transforming (3.2), we can find

$$G^+(-1) = \exp\frac{1}{\pi}\int_0^\infty\left\{\ln[q^2(t)+m^2(t)] + 4t\,\mathrm{arctg}\frac{m(t)}{q(t)}\right\}\frac{dt}{1+4t^2} \qquad (4.8)$$

$$G^+\left(-\frac{1}{2}\right) = \frac{\sqrt{2}}{\sqrt{3}\sin\alpha\cos\alpha/2}\exp\frac{1}{\pi}\int_0^\infty \mathrm{arctg}\frac{m(t)}{q(t)}\frac{dt}{t}$$

where

$$q(t) = \frac{1}{2\,\mathrm{ch}^2 t\pi}\Big\{\sin^2\alpha\,(\cos\alpha + \mathrm{ch}\,2\alpha t)\,[\mathrm{ch}\,2t\,(\pi-\alpha) - \cos\alpha] +$$
$$(t\sin 2\alpha + \cos\alpha\,\mathrm{sh}\,2\alpha t)\,[t\sin 2\alpha + \cos\alpha\,\mathrm{sh}\,2t\,(\pi-\alpha)] +$$
$$2\,(\cos\alpha\,\mathrm{ch}\,2\alpha t - \cos 2\alpha)\Big[\cos^2\frac{\alpha}{2}\,\mathrm{ch}^2 t\,(\pi-\alpha) -$$
$$\sin^2\frac{\alpha}{2}\,\mathrm{sh}^2 t\,(\pi-\alpha) - ({}^1\!/_4 - t^2)\sin^2\alpha\Big] +$$
$$\sin\alpha\,\mathrm{sh}\,2\alpha t\,[\sin\alpha\,\mathrm{sh}\,2t\,(\pi-\alpha) - 2t\sin^2\alpha]\Big\}$$

$$m(t) = \frac{1}{2\,\mathrm{ch}^2 t\pi}\Big\{\sin\alpha(\cos\alpha + \mathrm{ch}\,2\alpha t)[t\sin 2\alpha + \cos\alpha\,\mathrm{sh}\,2t(\pi-\alpha)] -$$
$$(t\sin 2\alpha + \cos\alpha\,\mathrm{sh}\,2\alpha t)\sin\alpha\,[\mathrm{ch}\,2t\,(\pi-\alpha) - \cos\alpha] +$$
$$2\sin\alpha\,\mathrm{sh}\,2\alpha t\Big[\cos^2\frac{\alpha}{2}\,\mathrm{ch}^2 t\,(\pi-\alpha) - \sin^2\frac{\alpha}{2}\,\mathrm{sh}^2 t\,(\pi-\alpha) -$$
$$({}^1\!/_4 - t^2)\sin^2\alpha\Big] - (\cos\alpha\,\mathrm{ch}\,2\alpha t - \cos 2\alpha)\times$$
$$[\sin\alpha\,\mathrm{sh}\,2t\,(\pi-\alpha) - 2t\sin^2\alpha]\Big\}$$

The dependence of the dimensionless length d_* (equal to $\tau_s^2 d/K_I^2$) on the angle α determined on a computer by formulas (4.7) and (4.8) is shown in Fig. 3. As is seen, this dependence has a maximum for $\alpha_* = 72°$ (the error does not exceed 1°); this maximum equals 0.046. It is natural to assume that the slip lines near the tip of a crack in a homogeneous and isotropic body (in strength), develop in the direction of this maximum, i.e. at the angle $\alpha_* = 72°$ to the continuation of the crack. The length of the plastic segments hence equals

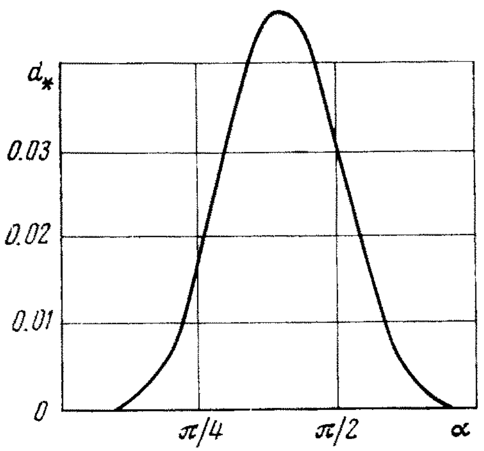

Fig. 3

$$d = 0.046 K_1^2/\tau_s^2 \qquad (4.9)$$

It should be noted that these results are quite close to the numerical results [4] obtained by means of flow theory for an ideal elastic-plastic material with a Mises condition by the finite element method. We recall that according to their analysis for $\nu = 0.3$, the maximum spacing of the spread plastic "ears" from the tip of the crack equals approximately $0.044\ K_1^2\ \tau_s^{-2}$, where the deviation of the corresponding maximum radius vector from the crack direction is approximately 70°.

The considered problem is easily generalized by taking account of the homogeneous side tension of the body along the crack by an arbitrary stress σ_x. For ideally brittle bodies this tension does not influence the local asymptotics of the field near the tip of the crack. This influence is easily computed in this problem. To do this it is sufficient to add the tension field σ_x to the perturbed field which is evidently obtained from the considered solution, if the quantity τ_s is replaced everywhere therein by $\tau_s + {}^1\!/_2\,\sigma_x \sin 2\alpha$.

In particular, according to (4.7), the length of the slip band is equal

$$d = \frac{\pi}{32} \frac{K_I^2 \sin^2 \alpha \cos^2 \alpha / 2 [G^+(-1)]^2}{[G^+(-1/2)]^2 (\tau_s + 1/2\, \sigma_x \sin 2\alpha)^2} \tag{4.10}$$

As is seen, other conditions being equal, the side tension diminishes the size of the plastic slip band.

Let us calculate the opening of the crack $2v_0$ at its tip for $\alpha = \alpha_* = 72°$. By using (2.10), (2.12), (3.10), we find

$$2v_0 = 2 \sin \alpha_* \int_0^d \left[\frac{\partial u_r}{dr}\right] dr = \frac{16(1-\nu^2) K_I^2}{E\tau_s} \frac{\sin \alpha_* G^-(0)}{\pi G^+(-1)} d_*(\alpha_*) = \tag{4.11}$$

$$0.2228 \frac{(1-\nu^2) K_I^2}{E\tau_s} \frac{G^-(0)}{G^+(-1)}$$

$$G^-(0) = \exp\left\{-\frac{1}{\pi} \int_0^\infty \left\{\ln [q^2(t) + m^2(t)] - 4t \operatorname{arctg} \frac{m(t)}{q(t)}\right\} \frac{dt}{1+4t^2}\right\}$$

Evaluating the integral on a computer, we obtain

$$2v_0 = 0.2222 \frac{(1-\nu^2) K_I^2}{E\tau_s'} \tag{4.12}$$

In particular, for $\nu = 0.3$ we have

$$2v_0 = 0.202 \frac{K_I^2}{E\tau_s} \tag{4.13}$$

which almost agrees with the corresponding numerical solution [4] for the spread plastic zone (in this solution the factor in (4.13) equals 0.21). Taking into account the side tension σ_x, formula (4.12) can be written as follows:

$$2v_0 = 0.2222 \frac{(1-\nu^2) K_I^2}{E(\tau_s' + 1/2\, \sigma_x \sin 2\alpha_*)} \tag{4.14}$$

By assumption, the size of the plastic domain is quite small; therefore, the beginning of crack development is determined by the concept of K_{Ic}: as soon as the quantity K_I equals the fracture ductility of the material K_{Ic} the crack starts to grow. According to (4.14), the magnitude of the critical opening of the crack (C.O.D.) will hence not be a constant of the material since it will still depend on the external load σ_x. The result presented shows therefore that the widely used criterion C.O.D. is not universal and not local.

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DYNAMIC DEFORMATION OF QUASI-ISOTROPIC COMPOSITE MEDIA

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The problem of a macroscopic description of the dynamics of an elastic composite medium without using assumptions concerning the uniformity of the mean stress-strain states, the magnitudes of the fluctuations and the statistics of the medium parameters [1] is considered. Operator relationships between the mean stress and strain fields allow the application of operator algebra which is well developed in creep theory [2]. It is shown that a quasistatic (matrix) part of the elastic operators, calculated by the method of replacement of variables, yields exact values of the elastic moduli of the composite, where the equations obtained are analogous to the self-consistent field equations [3 — 5]. The interrelation between the mean dimensions of the inhomogeneities and the lengths of the incident and scattered waves is investigated for a specific correlation function.

Analytical computational formulas for the elastic moduli of composite media have been obtained in [3 — 5] on the basis of classical solutions and the self-consistent field method. Exact formulas for the stochastic model have first been obtained in [6] on the basis of a strongly isotropic model, and on the basis of an equivalent singular approximation in [7]. The derivation of exact formulas requires homogeneity of the mean stress-strain states and summation of infinite sequences of the perturbation series (operators, in the general case) in the cases considered.

The method of replacing the field variables by their polarized values turns out to be equivalent to a partial summation of definite kinds of Feynman diagrams. It is established by a direct computation on the basis of the compatibility equations that the macroscopic moduli determined by the equations obtained are exact. A direct analytical comparison with formulas presented in [7, 8] is possible for two-phase composites; numerical computations for certain polycrystals of cubic symmetry yield the same values as the formulas in [6].